On $(K_q; k) -$ stable Graphs

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Introduction

In this project, we explore the results of the article "On $(K_q;k)$-Stable Graphs" by Andrzej Żak [4]. Let k be any nonnegative integer, and let G, H denote any two finite, simple graphs. We say that G is $(H;k)$-*stable* if, with the removal of any k vertices from G, there remains a subgraph of G that is isomorphic to H. If G is an $(H;k)-$stable graph with the fewest possible edges, then the number of edges of G is denoted $\operatorname{stab}(H;k)$. In this paper, Żak considers the case when H is the complete graph, K_q, on q vertices where q is any positive integer. He shows that, with the exception of some small values of k,

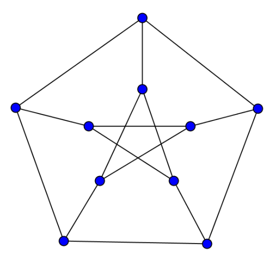
$$\operatorname{stab}(K_q;k) = (2q-3)(k+1);$$

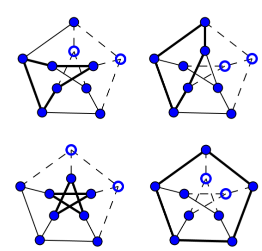
confirming a conjecture of Dudek et al [3]. Furthermore, he characterizes the extremal graphs that meet this bound.

Graphs

A graph G consists of a finite vertex set $V(G)$, and edge set $E(G)$ where $E(G)$ is a collection of 2-element subsets (the *edges*) of $V(G)$. When $\{x,y\} \in E(G)$ we write $x \sim y$. A vertex that does not belong to any edge is called an *isolated* vertex. Let the *order* of G be denoted $|G| := |V(G)|$ and let the *size* of G be denoted $\|G\| := |E(G)|$. Two graphs G, H are isomorphic if there is a bijection $f: V(G) \to V(H)$ such that $u \sim v$ if and only if $f(u) \sim f(v)$. Let H be any graph and k a non-negative integer. A graph G is called $(H;k)-$*vertex stable* or $(H;k)-$*stable* if G contains a subgraph isomorphic to H even after removing any k of its vertices.

Example 1 The Petersen Graph as a $(C_5;2)-$stable graph.





The Petersen Graph

Illustration of the Petersen Graph as a $(C_5;2)-$stable graph.

If G has the minimum size of any $(H;k)$-stable graph, then $\|G\| = \operatorname{stab}(H;k)$, and we refer to G as a *minimum* $(H;k)$-stable graph. Note that if H does not have isolated vertices, then after adding to or removing from a $(H;k)-$stable graph any number of isolated vertices, we still have an $(H;k)-$stable graph with the same size. Thus, we will assume that no such minimum (H,k)-stable graph has isolated vertices.

General Bounds

Lemma 1: If G is a minimum $(H;k)-$stable graph, then every vertex and every edge of G belongs to some subgraph of G isomorphic to H.

Proof:

Suppose there is an edge $e \in E(G)$ which is not in any subgraph of G isomorphic to H. Then $G-e$ is still $(H;k)-$stable with a smaller size than G, which is a contradiction since we were assuming G is minimum. If there exists a vertex v which is not in any subgraph of G isomorphic to H, then the same is true for each edge containing v. This again contradicts the minimality of G. ■

The following proposition will give a lower bound on the vertex degrees in G, and will be used frequently throughout the paper.

Proposition 1: Let δ_H be the minimum degree of a graph H. Then in any minimum $(H;k)-$stable graph G, $d_G(v) \geq \delta_H$ for each vertex $v \in G$.

Proof:

Let G be minimum $(H;k)-$stable, and let H_G be any subgraph of G isomorphic to H. Now assume, by way of contradiction, that there exists $v_0 \in G$ such that $d(v_0) < \delta_H$. Note, then, that $v_0 \notin V(H_G)$. Then by Lemma 1 we have that G is not minimum, which is a contradiction. ■

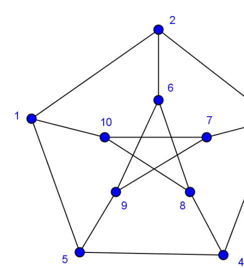
Theorem 2: If G is any minimum $(H;k)-$stable graph, then

$$|G| - \delta_H \sum_{v \in V(G)} \frac{1}{d_G(v)+1} \geq k+1.$$

Moreover, if G is not a union of cliques, then the inequality is strict.

Proof:

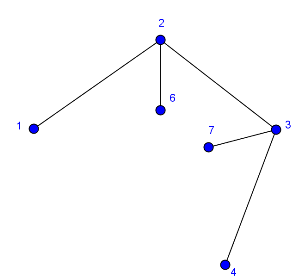
First note that for all $v \in V(G)$, by Proposition 1 we have that $d_G(v) \geq \delta_H$. Let $\deg_\sigma^-(v)$ denote the number of neighbors of v that are on the left of v in ordering σ, where σ is any ordering of the vertices of graph G. Then let S_σ denote the set of all vertices v with $\deg_\sigma^-(v) \leq \delta_H - 1$. What we want to do next is remove from G the set of vertices in $V(G) \setminus S_\sigma$, inducing a subgraph on S_σ that will contain no copies of H. This will eventually lead to a bound on k. To fully appreciate the strategy that is being used here, let's look at an example. Consider our previous example of the Petersen graph, and recall that it was $(C_5, 2)$-stable. Suppose σ is the ordering of the vertices given below.



Generating a table that tracks the values of $\deg_\sigma^-(v_n)$, we have the following:

n	1	2	3	4	5	6	7	8	9	10
$\deg_\sigma^-(v_n)$	0	1	1	1	2	1	1	2	3	3

Observe that $S_\sigma := \{v_i \mid \deg_\sigma^-(v_i) \leq 1\} = \{v_1, v_2, v_3, v_4, v_6, v_7\}$. Hence $V(G) \setminus S_\sigma := \{v_5, v_8, v_9, v_{10}\}$. Note that removing $V(G) \setminus S_\sigma$ from G yields the following graph:



Thus we have destroyed all copies of C_5. This strategy is clever, in the sense that it is designed to "pick on" the vertices that have the most "left sided" neighbors of any random permutation. This type of ordering essentially gives us a way to assign an "weight" to a vertex based on how many neighbors it has to its left in the table. The higher that weight is, the better candidate it is to remove since each left-neighbor represents an adjacency. Hence by removing all the vertices with high enough weights (weights $\geq \delta_H$) we ensure the fact that no copies of H will remain once $V(G) \setminus S_\sigma$ is removed. For example, the vertices we are left with in the above example are the vertices that, in the ordering, had only one left-neighbor.

In general, with any ordering we destroy all copies of H by consecutively eliminating all vertices of $V(G) \setminus S_\sigma$. Each vertex in S_σ has left degree $\leq \delta_H - 1$ and thus cannot be the rightmost vertex of any copy of H in the induced subgraph. We started with the assumption that G is $(H;k)-$stable, thus we get that $|G| - |S_\sigma| \geq k+1$ for any ordering σ.

Lemma 2 (Caro and Wei): Let $\deg_\sigma^-(v)$ denote the number of neighbors of v that are on the left of v in ordering σ, where σ is any ordering of $V(G)$. Let S_σ denote the set of all vertices v with $\deg_\sigma^-(v) \leq \delta_H - 1$. If $\delta_H = 1$, then each set S_σ is an independent set.

Proof:

Note that $S_\sigma := \left\{ v_i \mid \deg_\sigma^-(v_i) \leq 0 \right\}$. So in the ordering σ, each vertex $v \in S_\sigma$ has no left neighbors. Therefore, S_σ is an independent set. ■

We are interested in finding a way to approximate the size of S_σ for any ordering σ. To do this, we will implement a method of using an appropriate expected value to capture a lower bound on the size of S_σ.

To further understand the probability we are looking for, let us pick apart an arbitrary ordering of our vertex set of size n and generate a probability to measure the likelihood that any arbitrary v is an element of S_σ. We calculate the probability that, under the ordering σ, v has at most i neighbors to its left. We begin with the choose function

$$\binom{n}{d_G(v)+1},$$

by selecting $d_G(v)+1$ slots of the n slots available for placing the vertices.

$$\begin{array}{ccccccccccccc} \cdots & \_ & \cdots & \_ & \cdots & \_ & \cdots & \_ & \cdots & \_ & \cdots & \_ & \cdots \\ & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & \end{array}$$

$d_G(v)+1$ slots of the n slots available for placing the vertices are chosen.

We now have enough locations to place all v and all of v's neighbors. From here we select the first $i+1$ slots. Note that v can exist in any of these slots and still satisfy our condition that $\deg_\sigma^-(v) \leq i$.

$$\begin{array}{ccccccccccccc} \cdots & \_ & \cdots & \_ & \cdots & \_ & \cdots & v & \cdots & \_ & \cdots & \_ & \cdots \\ & \uparrow & & \uparrow & & \uparrow & & & & \uparrow & & \uparrow & \end{array}$$

The vertex v can exist in any of the first $i+1$ slots.

At this point we now have the following:

$$\binom{n}{d_G(v)+1}(i+1).$$

From here we need to arrange the remaining vertices, namely the neighbors of v and the non-neighbors of v. The number of neighbors that v has is determined by its degree. Thus let $N_G(v) := \{v_i \in V(G) \mid v_i \sim v, v \neq v_i\}$. Then $|N_G(v)| = d_G(v)$ and the number of ways to permute $N_G(v)$ is $(d_G(v))!$. Since we have that v and all of v's neighbors have been arranged in the ordering, we have to place the remaining vertices which is n vertices minus 1 for having placed v and minus $d_G(v)$ for having already placed all of v neighbors, hence placing the remaining vertices yields $(n - d_G(v) - 1)!$ permutations. Thus we have the following:

$$\left(\begin{array}{c} n \\ d_G(v) + 1 \end{array} \right) (i + 1) (d_G(v))! \, (n - d_G(v) - 1)!,$$

and have successfully counted all of the permutations where our analyzed vertex has i neighbors to its left under the ordering of σ. Since we are interested in the probability of this happening, we divide our counted value by the total number of ways to order n vertices, which is $n!$. Hence we get that

$$Pr\left(\deg_{\sigma}^{-}(v) \leq i\right) = \frac{\left(\begin{array}{c} n \\ d_G(v) + 1 \end{array} \right) (i + 1) (d_G(v))! \, (n - d_G(v) - 1)!}{n!}.$$

Reducing this quotient we get:

$$Pr\left(\deg_{\sigma}^{-}(v) \leq i\right) = \frac{i + 1}{d_G(v) + 1},$$

Note that for our construction of S_σ we have that $i = \delta_H - 1$, thus

$$\frac{i + 1}{d_G(v) + 1} = \frac{\delta_H}{d_G(v) + 1}.$$

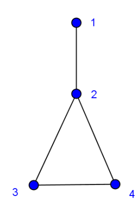
We get from this that

$$Pr(v \in S_\sigma) = \frac{\delta_H}{d_G(v) + 1}.$$

Then by Linearity of Expectation,

$$E(S_\sigma) = \sum_{v \in V(G)} \frac{\delta_H}{d_G(v) + 1}.$$

Example 2 of Expected Value; consider the paw graph and all of the permutations on a fixed labeling of its vertices:



Paw Graph

1234	2134	3124	4123
1243	2143	3142	4132
1342	2314	3214	4213
1324	2341	3241	4231
1432	2413	3412	4312
1423	2431	3421	4321

Permutations on the vertex set $\{1, 2, 3, 4\}$.

Let σ be a random permutation of the vertices of the Paw Graph. Let $i = 1$ and set $\deg_{\sigma}^{-}(v) \leq i$ where $\deg_{\sigma}^{-}(v)$ is set to equal the number of left neighbors of v under the ordering of σ. Furthermore, let $S_\sigma := \left\{v \mid \deg_{\sigma}^{-}(v) \leq i\right\}$. Note that in the above table of all 24 permutations on the vertex set of the paw graph, 20 of them have an ordering such that $|S_\sigma| = 3$, and 4 of them have an ordering such that $|S_\sigma| = 2$. Then there are $(20 \cdot 3 + 4 \cdot 2)$ vertices out of the 24 permutations on the vertex set that satisfy $\deg_\sigma^-(v) \leq i$, or 68/24. Note that if we calculate the expectation from the formula we get:

$$\begin{aligned}
E(S_\sigma) &= \sum_{v \in V(G)} \frac{i+1}{d_G(v)+1} \\
&= \frac{2}{1+1} + \frac{2}{2+1} + \frac{2}{2+1} + \frac{2}{3+1} \\
&= \frac{24}{24} + \frac{16}{24} + \frac{16}{24} + \frac{12}{24} \\
&= \frac{68}{24}.
\end{aligned}$$

Now consider all the possibilities of how $|S_\sigma|$ relates to $\sum_{v \in V(G)} \frac{\delta_H}{d_G(v)+1}$, and furthermore, how it affects $|G| - \delta_H \sum_{v \in V(G)} \frac{1}{d_G(v)+1} \geq k+1$. Note that if there exists such an ordering where $|S_\sigma| < \sum_{v \in V(G)} \frac{\delta_H}{d_G(v)+1}$, then there must exist an ordering σ' such that $|S_{\sigma'}| > \sum_{v \in V(G)} \frac{\delta_H}{d_G(v)+1}$, because the expectation is exactly equal to $\sum_{v \in V(G)} \frac{\delta_H}{d_G(v)+1}$. This implies that for any ordering σ we can get that $|G| - |S_\sigma| > |G| - \delta_H \sum_{v \in V(G)} \frac{1}{d_G(v)+1} \geq k+1$. Note also that $|G| - \delta_H \sum_{v \in V(G)} \frac{1}{d_G(v)+1} = k+1$ if for every ordering of σ, $|S_\sigma|$ is the same. Let us further consider the scenario where for any ordering of σ, $|S_\sigma|$ is the same.

Let C be any component of G and let $v \in V(C)$. Let $\delta = \delta_H$ and let $|S_\sigma| = |S_{\sigma'}|$ on any ordering of σ. Consider the following ordering σ of vertices of C:

$$v_1, v_2, \ldots, v_\delta, v_{\delta+1}, v_{\delta+2}, \ldots, v_{|C|}.$$

Let $v_{\delta+1} = v$ and $v_1, \ldots, v_\delta$ be neighbors of v. Note that $v_1, \ldots, v_\delta$ need not necessarily be all the neighbors of v since it was assumed that $v \in V(C)$ and by Proposition 1 we have that for any $v \in V(G)$, $d_G(v) \geq \delta_H$. From this ordering on the vertex set we get that $\deg_\sigma^-(v_{\delta+1}) = \delta$. Now consider the ordering σ' of vertices of C:

$$v_{\delta+1}, v_1, v_2, \ldots, v_\delta, v_{\delta+2}, \ldots, v_{|C|}.$$

By assumption we have that $|S_\sigma| = |S_{\sigma'}|$, thus we get that $v_{\delta+1} \in S_{\sigma'}$, $v_\delta \notin S_{\sigma'}$ and that $\deg_\sigma^-(v_\delta) = \delta$. This process can be repeated producing the following:

$$\begin{aligned}
v_\delta, v_{\delta+1}, v_1, v_2, \ldots, v_{\delta-2}, v_{\delta-1}, v_{\delta+2}, \ldots, v_{|C|} \implies \deg_\sigma^-(v_{\delta-1}) &= \delta \\
v_{\delta-1}, v_\delta, v_{\delta+1}, v_1, \ldots, v_{\delta-3}, v_{\delta-2}, v_{\delta+2}, \ldots, v_{|C|} \implies \deg_\sigma^-(v_{\delta-2}) &= \delta \\
v_{\delta-2}, v_{\delta-1}, v_\delta, v_{\delta+1}, \ldots, v_{\delta-3}, v_{\delta-2}, v_{\delta+2}, \ldots, v_{|C|} \implies \deg_\sigma^-(v_{\delta-3}) &= \delta \\
&\vdots \\
v_2, v_3, v_4, v_5, \ldots, v_{\delta+1}, v_1, v_{\delta+2}, \ldots, v_{|C|} \implies \deg_\sigma^-(v_1) &= \delta.
\end{aligned}$$

Thus we get that the vertices $v_1, \ldots, v_{\delta+1}$ induce a clique. Note that v and its neighbors were chosen arbitrarily, hence $\{v\} \cup N_G(v)$ induce a clique for each $v \in V(C)$, it follows that C is a clique. ■

Lemma 2: Pick any $r > 0$ and $\ell \in \mathbb{N}$. The expression $\sum_{j=1}^{\ell} \frac{1}{x_j}$ with $\sum_{j=1}^{\ell} x_j = r$ and $x_j > 0$ is minimal if all the x_j are equal.

Proof:

Let $f(x_1, \ldots, x_n) = \sum_{k=1}^{n} \left(\frac{1}{x_k}\right)$ and let $g(x_1, \ldots, x_n) = \sum_{k=1}^{n} x_k = k$ where $x_1, \ldots, x_n$ are non-zero numbers and $k > 0$. Let f_{x_i} be used to denote the partial derivative of f with respect to x_i, and similarly for g. Using methods of multivariate constrained optimization, we can minimize our variables. Note when solving for the Lagrange multipliers:

$$\begin{aligned}
f_{x_1}(x_1,\ldots,x_n) &= \lambda g_{x_1}(x_1,\ldots,x_n)\\
x_1^{-2} &= \lambda\\
f_{x_2}(x_1,\ldots,x_n) &= \lambda g_{x_2}(x_1,\ldots,x_n)\\
x_2^{-2} &= \lambda\\
&\vdots\\
f_{x_n}(x_1,\ldots,x_n) &= \lambda g_{x_2}(x_1,\ldots,x_n)\\
x_n^{-2} &= \lambda,
\end{aligned}$$

we get that $\lambda = x_1^{-2} = x_2^{-2} = \cdots = x_n^{-2}$. We assumed that $x_1,\ldots,x_n$ are non-zero numbers, then the equality only holds when $x_1 = x_2 = \cdots = x_n$. ■

Corollary 3: Let H be any graph and let δ_H denote the minimum degree of H. Then

$$\operatorname{stab}(H;k) \geq (k+1)\left(\delta_H + \sqrt{\delta_H(\delta_H - 1)} - \frac{1}{2}\right).$$

Proof:

The corollary implies that we can find a lower bound on our minimum stable graphs. To illustrate this let G be a minimum $(H;k)$ – stable graph. Let δ_H denote the minimum degree of H and let $d_G = \frac{2\|G\|}{|G|}$, the average degree of G.

From Lemma 2 we gain the following inequality:

$$\begin{aligned}
\sum_{v\in V(G)} \frac{1}{d(v)+1} &\geq \sum_{v\in V(G)} \frac{1}{d_G+1}\\
&= \frac{|G|}{d_G+1}.
\end{aligned}$$

Using this inequality and Theorem 2 we get the following:

$$|G| \geq \delta_H \sum_{v\in V(G)} \frac{1}{d(v)+1} + k + 1 \geq |G|\frac{\delta_H}{d_G+1} + k + 1.$$

It follows that:

$$\|G\| = \left(\frac{d_G}{2}\right)|G| \quad \geq \quad \frac{(k+1)}{2}\cdot\frac{d_G(d_G+1)}{(d_G+1-\delta_H)}.$$

Let f represent the characteristic polynomial of $\frac{d_G(d_G+1)}{(d_G+1-\delta_H)}$, in other words, let $x = d_G$ and consider the critical point of the function for when f is at its minimum.

$$\begin{aligned}
f(x) &= \frac{x(x+1)}{(x+1-\delta_H)}\\
f'(x) &= \frac{(x+1-\delta_H)(2x+1) - (x(x+1))}{(x+1-\delta_H)^2}
\end{aligned}$$

Note that we are interested in when $f'(x) = 0$, which only happens when the numerator is equal to 0. This yields a quadratic polynomial:

$$\begin{aligned}
(x+1-\delta_H)(2x+1) - (x(x+1)) &= 0\\
x^2 + 2x(1-\delta_H) + (1-\delta_H) &= 0
\end{aligned}$$

Using the quadratic formula to solve for the roots of the polynomial yields:

$$x = (-1+\delta_H) \pm \sqrt{\delta_H(\delta_H - 1)}.$$

Hence we get that $x_0 = -1+\delta_H+\sqrt{\delta_H(\delta_H-1)}$. Plugging x_0 back into f and simplifying, we get:

$$\begin{aligned} f(x_0) &= \frac{x_0(x_0+1)}{(x_0+1-\delta_H)} \\ &= 2\left(\delta_H+\sqrt{\delta_H(\delta_H-1)}-\frac{1}{2}\right). \end{aligned}$$

Then it follows that

$$\|G\| = \left(\frac{d_G}{2}\right)|G| \geq \frac{(k+1)}{2}\cdot\frac{d_G(d_G+1)}{(d_G+1-\delta_H)} \geq (k+1)\left(\delta_H+\sqrt{\delta_H(\delta_H-1)}-\frac{1}{2}\right).$$

Since we had that G was a minimum $(H;k)-$stable graph, then $\|G\| = \mathrm{stab}(H;k)$, and

$$\mathrm{stab}(H;k) \geq (k+1)\left(\delta_H+\sqrt{\delta_H(\delta_H-1)}-\frac{1}{2}\right). \blacksquare$$

Thus we have completed a general lower bound of the size of a graph G when it is minimum $(H;k)-$stable.

Complete Graphs, K_q

With the use of Theorem 2 and Corollary 3, we can start establishing bounds and results for specific graphs, namely, the complete graphs K_q.

Theorem 4: Let G be a $(K_q;k)-$stable graph, $q\geq 2$ and $k\geq 0$. Then

$$\|G\| \geq (2q-3)(k+1),$$

with equality if and only if G is a disjoint union of cliques K_{2q-3} and K_{2q-2}.

Proof:

Let G be a $(K_q;k)-$stable graph, $q\geq 2$ and $k\geq 0$. Note that $\delta_H = (q-1)$ then let $x_{t+} = q-1+\sqrt{(q-1)(q-2)}$ and $x_{t-} = q-3+\sqrt{(q-1)(q-2)}$. Consider

$$f'(x_{t+}) = \frac{(x_{t+})^2+2(x_{t+})(1-\delta_H)+(1-\delta_H)}{((x_{t+})+1-\delta_H)^2},$$

which simplifies to

$$f'(x_{t+}) = \frac{2\sqrt{(q-1)(q-2)}+1}{\left(\sqrt{(q-1)(q-2)}+1\right)^2}.$$

Note that $f'(x_{t+})>0$ for $q>2$, then by the first derivative test, f is increasing for $x\geq x_0$. Also consider

$$f'(x_{t-}) = \frac{(x_{t-})^2+2(x_{t-})(1-\delta_H)+(1-\delta_H)}{((x_{t-})+1-\delta_H)^2},$$

which simplifies to

$$f'(x_{t-}) = \frac{1-2\sqrt{(q-2)(q-1)}}{\left(\sqrt{(q-2)(q-1)}+1\right)^2}.$$

Which for values of $q > 2$, $f'(x_{t-}) < 0$, thus implying by the first derivative test that f is decreasing for $x \leq x_0$. To establish the bounds on x_0 consider a construction of x_0 under the constraint that $x_0 = q - 1 + \sqrt{(q-1)(q-2)} - 1$ is when f is at its minimum. Then:

$$\begin{array}{ccccc}
(q-2) & \leq & (q-2) & \leq & (q-1) \\
(q-2)(q-1) & \leq & (q-2)(q-1) & \leq & (q-1)^2 \\
(q-2)^2 & \leq & (q-2)(q-1) & \leq & (q-1)^2 \\
(q-2) & \leq & \sqrt{(q-2)(q-1)} & \leq & (q-1) \\
(q-2)+q-2 & \leq & \sqrt{(q-2)(q-1)}+q-2 & \leq & (q-1)+q-2 \\
2q-4 & \leq & \sqrt{(q-2)(q-1)}+q-2 = x_0 & \leq & 2q-3
\end{array}$$

The bounds for x_0 are consecutive integers, moreover we have

$$\begin{aligned}
f(2q-4) &= \frac{(2q-4)(2q-4+1)}{2q-4+1-(q-1)} \\
&= 2(2q-3)
\end{aligned}$$

$$\begin{aligned}
f(2q-3) &= \frac{(2q-3)(2q-3+1)}{2q-3+1-(q-1)} \\
&= 2(2q-3)\,.
\end{aligned}$$

Since our characteristic function was derived for when $x = d_G$, it follows that the lower bound on $\|G\|$ is achieved when $d_G \in [2q-4, 2q-3]$. Since we have that the expression $\sum_{j=1}^{l} \frac{1}{x_j}$ with $\sum_{j=1}^{l} x_j = const$, $x_j > 0$ is minimal if all the x_j are equal, and d_G as the average degree of G; then we want the degree of any vertex of G to not deviate too far from d_G, thus minimizing the sum. By assumption we have that G is a $(K_q; k) -$ stable graph, so it is safe to assume that $d_G(v) \in \{2q-4, 2q-3\}$. It can be shown then that the equality of $\|G\| \geq (2q-3)(k+1)$ is strong if and only if G is a disjoint union of cliques K_{2q-3} and K_{2q-2}. For this let m denote the number of vertices of G with degree equal to $2q-3$. Note that in corollary 3 the following was shown:

$$|G| \geq \delta_H \sum_{v \in V(G)} \frac{1}{d(v)+1} + k + 1 \geq |G| \frac{\delta_H}{d_G+1} + k + 1.$$

Consider the latter part of the inequality and drop the common factor δ_H, then its left to show:

$$\sum_{v \in V(G)} \frac{1}{d(v)+1} \geq |G| \frac{1}{d_G+1}.$$

Note that $|G| = |G| - m + m$, then considering the right hand side of the equation:

$$\begin{aligned}
|G| \frac{1}{d_G(v)+1} &= (|G| - m + m) \frac{1}{d_G(v)+1} \\
&= m \frac{1}{d_G(v)+1} + (|G| - m) \frac{1}{d_G(v)+1}.
\end{aligned}$$

Because m denotes the number of vertices of G with degree equal to $2q-3$, then

$$m \frac{1}{d_G(v)+1} + (|G| - m) \frac{1}{d_G(v)+1} = m \frac{1}{(2q-3)+1} + (|G| - m) \frac{1}{(2q-4)+1}.$$

Suppose that $d_G(v) \in \{2q-4, 2q-3\}$ for *every* $v \in V(G)$, thus we obtain for $d_G(v) = 2q-4$, $m = 0$ and:

$$\begin{aligned}
m \frac{1}{d_G(v)+1} + (|G| - m) \frac{1}{d_G(v)+1} &= |G| \frac{1}{(2q-4)+1} \\
&= \sum_{v \in V(G)} \frac{1}{(2q-4)+1}.
\end{aligned}$$

And for $d_G(v) = 2q-3$ for *every* $v \in V(G)$, $m = |G|$ and:

$$\begin{aligned} m\frac{1}{d_G(v)+1} + (|G|-m)\frac{1}{d_G(v)+1} &= |G|\frac{1}{(2q-3)+1} \\ &= \sum_{v \in V(G)} \frac{1}{(2q-3)+1}. \end{aligned}$$

Now let $0 < m < |G|$ where the vertex set can be partitioned into two sets, one of vertices of degree equal to $2q-3$, and one set of vertices of degree equal to $2q-4$. Then $m = |\{v \mid d_G(v) = 2q-3\}|$ and $|G| - m = |\{v \mid d_G(v) = 2q-4\}|$

$$\begin{aligned} m\frac{1}{(2q-3)+1} + (|G|-m)\frac{1}{(2q-4)+1} &= \sum_{i=1}^{m} \frac{1}{(2q-3)+1} + \sum_{i=1}^{|G|-m} \frac{1}{(2q-4)+1} \\ &= \sum_{i=1}^{|G|} \frac{1}{(d_G(v))+1} \\ &= \sum_{v \in V(G)} \frac{1}{(d(v))+1}. \end{aligned}$$

To prove the other direction of the "if and only if", suppose

$$\sum_{v \in V(G)} \frac{1}{d(v)+1} = m\frac{1}{2q-2} + (|G|-m)\frac{1}{2q-3} = \frac{2(q-1)|G|-m}{2(q-1)(2q-3)}.$$

Then

$$\begin{aligned} \sum_{v \in V(G)} \frac{1}{d(v)+1} &= \frac{2(q-1)|G|-m}{2(q-1)(2q-3)} \\ &= m\frac{1}{(2q-3)+1} + (|G|-m)\frac{1}{(2q-4)+1}. \end{aligned}$$

Previously it was shown that the equality held if $d_G(v) \in \{2q-4, 2q-3\}$ for *every* $v \in V(G)$. Suppose that $d_G(v) \in \{2q-4, 2q-3\}$ for some $v \in V(G)$ and suppose there exist $v_c \in V(G)$ such that $d_G(v_c) = \epsilon \notin \{2q-4, 2q-3\}$. Note that $|G| - m \neq 0$ and by assumption we have $d_G(v) > 2$. Note that $\frac{1}{d_G(v_c)+1} > 0$.

$$\begin{aligned} \sum_{v \in V(G)} \frac{1}{d(v)+1} &= m\frac{1}{2q-2} + (|G|-m)\frac{1}{2q-3} + \frac{1}{d_G(v_c)+1} \\ &\geq m\frac{1}{2q-2} + (|G|-m)\frac{1}{2q-3}. \end{aligned}$$

Which contradicts the assumption that the equality held. From this we get that

$$\sum_{v \in V(G)} \frac{1}{d(v)+1} \geq m\frac{1}{2q-2} + (|G|-m)\frac{1}{2q-3} = \frac{2(q-1)|G|-m}{2(q-1)(2q-3)},$$

with equality if $d_G(v) \in \{2q-4, 2q-3\}$ for every $v \in V(G)$. Which, in turn, yields the following:

$$|G| \geq \delta_H \sum_{v \in V(G)} \frac{1}{d_G(v)+1} + k + 1 \geq \delta_H \left(m\frac{1}{2q-2} + (|G|-m)\frac{1}{2q-3} \right) + k + 1 = \delta_H \left(\frac{2(q-1)|G|-m}{2(q-1)(2q-3)} \right) + k + 1.$$

Given that H is K_q, and $\delta_H = (q-1)$. The above implies

$$|G| - \frac{(q-1)|G|-m}{(2q-3)} \geq k+1.$$

From this and some algebra, it follows that

$$|G| \geq (k+1)\binom{2q-3}{q-2} - \frac{m}{2(q-2)}.$$

Now consider $\|G\|$. It has been established that G is minimal if $d_G(v) \in \{2q-4, 2q-3\}$. Once again let m denote the number of vertices of G with degree equal to $2q-3$. Then $m = |\{v \mid d_G(v) = 2q-3\}|$, $|G| - m = |\{v \mid d_G(v) = 2q-4\}|$, and

$$\begin{aligned}
\|G\| &= \binom{d_G}{2}|G| \\
&\geq \frac{m(2q-3)}{2} + \frac{(|G|-m)(2q-4)}{2} \\
&\geq \frac{m(2q-3) + \left(\left((k+1)\binom{2q-3}{q-2} - \frac{m}{2(q-2)}\right) - m\right)(2q-4)}{2} \\
&= (k+1)(2q-3),
\end{aligned}$$

as desired. ■

Frobenius Numbers

Given positive integers that are relatively prime, the Frobenius Number is the largest integer that cannot be obtained using linear combinations of the given integers. Note that the scalar multiples in the linear combination must also be positive. Closed form formulas exists for these numbers when dealing with strictly one positive integer, or two positive integers. There are no closed form solutions for $n > 2$ where n is the number of positive, relatively prime numbers given. A proof of the closed form formula for the case $n = 2$ is given as follows:

Lemma 3: Prove that if given positive integers n, m where $\gcd(n, m) = 1$, then the Frobenius Number for $ma + nb = K$ is $nm - n - m$.

Proof:

Without loss of generality assume $n > m$ and further suppose that $p = n - m$. Let $a, b \in \mathbb{N}$ and consider $ma + nb = K$. Note from substitution we get the following $ma + (m+p)b = K$, and from distributing and regrouping we get $m(a+b) + pb = K$. This formula will be used to analyze what integers can be generated with choices made for a, b.

Case 1: Let $b = 0$, then $m(a+0) + p(0) = ma$. Given a selection on a and m we can generate integers that look like am. This is not very useful since what we are looking for is a specific integer that once we step past that integer on the number line, am will be able to generate all of the following integers. Which is why am doesn't help us in identifying the Frobenius Number, because if it did, then all integers past a certain point would be expressible as the product of powers of a and m.

Case 2: Let $b = 1$, then $m(a+1) + p(1) = m(a+1) + p$. Note that $a + 1$ can take on any integer value except 0. So $m(a+1) + p$ can generate values $mk + p$ where $k \neq 0$, or namely, $m(a+1) + p$ cannot generate p itself.

Case 3: Let $b = 2$, then $m(a+2) + p(2) = m(a+2) + 2p$. Note now that $a + 2$ can take on any integer value except 0 and 1. So $m(a+2) + 2p$ can generate values $mk + 2p$ where $k \neq 0, 1$; or namely, $m(a+2) + 2p$ cannot generate $2p$, or $m + 2p$. At this point a pattern begins to emerge. Note that continuing this process of looking at each individual case, in the end we would end up looking at a total of m cases.

$\vdots$

Case m: Let $b = m-1$, then $m(a+b)+pb = m(a+(m-1))+p(m-1)$. Note that $a+(m-1)$ cannot take on value of $0, 1, \ldots, m-2$, and recall that $p = n-m$. Since we are interested in when K is not a solution of $m(a+(m-1))+p(m-1)$ we will substitute $a+(m-1)$ with our last value that $a+(m-1)$ cannot equal, namely $m-2$. Thus we get when simplifying the expression

$$\begin{aligned} K &= m(m-2)+p(m-1) \\ &= m(m-2)+(n-m)(m-1) \\ &= nm-n-m \end{aligned}$$

And thus we have the closed form formula for identifying the Frobenius Number for two relatively prime positive integers. ∎

Theorem 5: Let $q \geq 2$, $k \geq 0$ be non-negative integers. Then

$$\operatorname{stab}(K_q; k) \geq (2q-3)(k+1),$$

with equality if and only if $k = a(q-2)+b(q-1)-1$ for some non-negative integers a, b. In particular,

$$\operatorname{stab}(K_q; k) = (2q-3)(k+1) \text{ for } k \geq (q-3)(q-2)-1.$$

Furthermore, if G is a $(K_q; k)-$stable with $\|G\| = (2q-3)(k+1)$, then G is a disjoint union of cliques K_{2q-3} and K_{2q-2}.

Proof:

Let a, b be arbitrary, non-negative integers and define $G = aK_{2q-3} + bK_{2q-2}$. Then G is a disjoint union of cliques K_{2q-3} and K_{2q-2}. By Theorem 4 we have that $\|G\| = (2q-3)(k+1)$. Solving for k yields:

$$k = \frac{\|G\|}{(2q-3)} - 1.$$

Note that

$$\begin{aligned} \|G\| &= \|aK_{2q-3} + bK_{2q-2}\| \\ &= a\|K_{2q-3}\| + b\|K_{2q-2}\| \\ &= a\left(\frac{(2q-3)(2q-4)}{2}\right) + b\left(\frac{(2q-3)(2q-2)}{2}\right) \\ &= (2q-3)(a(q-2)+b(q-1)). \end{aligned}$$

Then

$$\begin{aligned} \frac{\|G\|}{(2q-3)} - 1 &= \frac{(2q-3)(a(q-2)+b(q-1))}{(2q-3)} - 1 \\ &= (a(q-2)+b(q-1)) - 1. \end{aligned}$$

Hence $k = (a(q-2)+b(q-1))-1$, thus $G = aK_{2q-3} + bK_{2q-2}$ is $(K_q; a(q-2)+b(q-1)-1)-$stable. Note that the Frobenius number can be identified for $\{q-2, q-1\}$, the given integers in the formula for k from G being $(K_q; a(q-2)+b(q-1)-1)-$stable. Let $m = q-2$ and $n = q-1$. The Frobenius Number K is given by the following formula: $nm-n-m = K$. Then

$$\begin{aligned} K &= nm-n-m \\ &= (q-1)(q-2)-(q-1)-(q-2) \\ &= (q-3)(q-2)-1. \end{aligned}$$

From Theorem 4 we get that $\|G\| \geq (2q-3)(k+1)$. Since it has been shown that $G = aK_{2q-3} + bK_{2q-2}$ *is* $(K_q; a(q-2) + b(q-1) - 1) - $*stable* it follows that stab $(K_q; k) \geq (2q-3)(k+1)$, with equality if and only if $k = a(q-2) + b(q-1) - 1$ for some non-negative integers a, b. Furthermore we get that k is bounded below by the Frobenius Number defined as $(q-3)(q-2) - 1$. ■

Future Work

Although much regarding the stability of graphs has been explored, especially regarding complete graphs, there is still much out there that has not been thoroughly established. As of the publication of this paper apart from some small value of q, the exact value of stab $(K_q; k)$ could be determined; stab $(K_7; 7)$ was the first value of q that was still open. For the most part only general bounds for the size of minimum stable graphs have been completely explored for families of graphs such as complete graphs and cycles, and a few bounds for paths have been worked out. Strong equalities for these graphs are rare to come by (most of which are currently where H is a complete graph) and providing such an equality or even a tighter bound on the size of a minimum stable graph could provide a rich area for research. Additionally, a question that arises is: if a strong equality can be established for stab $(H; k)$, is there a unique graph that is described by such an edge set? Currently, however, my personal interest lies in characterizing the minimum stable graphs for P_n.

Conclusion

In conclusion we have seen some general bounds for the size of minimal $(H; k) -$ stable graphs. Furthermore, with the exception of a few small values of q, the exact value of stab $(K_q; k)$ can be determined, and a lower bound for those few values for which the exact value does not exist.

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